Dual number quaternions as a mathematical tool to describe rigid body transformations

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Introduction: The formal description of rigid body transformations is the backbone of motion correction techniques in MRI since it both describes transformations within a given reference frame (i.e changes relative to an initial position) and transformations between reference frames (external tracking devices). The most common description is given in terms of 4x4 homogeneous matrices, where the matrix is composed of a 3x3 unitary rotation matrix and a displacement vector. A more compact form is given in terms of unit dual quaternions which are the algebraic representations of screw motions.

Theory: Dual numbers were invented by Clifford (1873) and are defined as $z = r + \epsilon d$ with the real and dual part *r* and *d* and the imaginary unit ϵ which has the property $\epsilon^2 = 0$. Dual number multiplication is associative but non-commutative ($z \times w \neq w \times z$). Dual quaternions are defined in analogy to dual numbers as $\mathbf{q} = \mathbf{r} + \epsilon\mathbf{d}$ where $\mathbf{r}$ and $\mathbf{d}$ are ordinary quaternions that can be used to describe the rotational and translational part. Rules for addition, multiplication, inversion etc. are derived. Dual quaternions

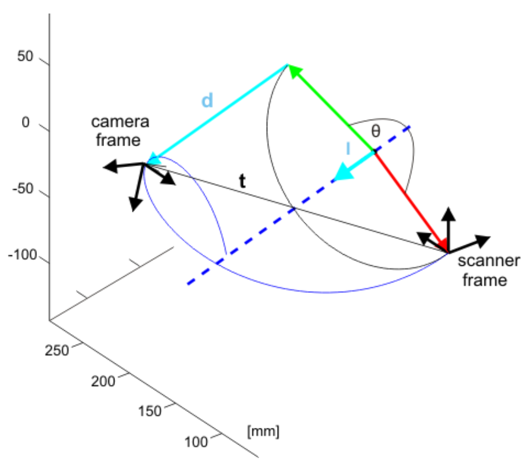

Figure 1 Geometry of a screw: every motion can be modeled as a rotation with angle θ about an axis at c, with direction l and a subsequent translation d along the axis.

$$\breve{\mathbf{q}} = \underbrace{\left(\cos\frac{\overset{\text{rotation angle}}{\theta}}{2}, \overset{\text{rotation axis}}{\mathbf{l}}\sin\frac{\theta}{2}\right)}_{\text{rotational unit quaternion}} + \epsilon\underbrace{\left(\frac{-\overset{\text{helix pitch}}{d}}{2}\sin\frac{\theta}{2}, \sin\frac{\theta}{2}\overset{\text{line moment}}{\mathbf{m}} + \frac{d}{2}\cos\frac{\theta}{2}\mathbf{l}\right)}_{\text{translation quaternion}}$$

are closely related to the geometry of a screw motion characterized by a rotation angle/axis, a line moment and a helix pitch (Fig.1). Dual quaternions allow to algebraically separate the angle and pitch information (which are constants of motion independent of the reference frame) from the rotation and displacement axis information.

Applications: Cross calibration of an external tracking system usually relies on solving the equation $\mathbf{A} = \mathbf{XBX}^{-1}$ for the unknown transformation $\mathbf{X}$ that connects motion in frames $\mathbf{A}$ and $\mathbf{B}$. Written as dual quaternions $\mathbf{a} = \mathbf{xbx}'$ it is possible to provide a one step solution for $\mathbf{x}$ which does not require non-linear minimization.

Conclusion: Unit dual quaternions provide a compact (8 components only) and efficient (e.g. inversion only involves sign changes) way to describe rigid body transformations. An immediate benefit is given by the ability to provide a one step solution to the cross calibration problem from at least 2 motions in both reference frames [1].

References:

[1] Konstantinos Daniilidis. Hand-eye calibration using dual quaternions. *The International Journal of Robotics Research*, 18(3):286–298, March 1999.

[2] Michael W. Walker, Lejun Shao, and Richard A. Volz. Estimating 3d location parameters using dual number quaternions. *Image Understanding*, 54:358–367, 1991.